Inside the eigenvalues of certain Hermitian Toeplitz band matrices

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Albrecht Böttcher and Egor Maximenko.

The $n \times n$ Toeplitz matrix $T_n(a)$ generated by a function a in L^1 on the complex unit circle $\mathbf{T}$ is defined by

$$T_n(a) = (a_{j-k})_{j,k=1}^n$$

$$a_\ell = \frac{1}{2\pi} \int_0^{2\pi} a(e^{ix}) e^{-i\ell x} dx \quad (\ell \in \mathbf{Z}).$$

The asymptotics of the eigenvalues of $T_n(a)$ as $n \to \infty$ has been thoroughly studied by many authors for now almost a century.
We here bound ourselves to the case where a is real-valued, in which case $\overline{a_\ell} = a_{-\ell}$ for all $\ell \in \mathbf{Z}$ and hence the matrices $T_n(a)$ are all Hermitian. The eigenvalues are then real and may be labeled so that

$$\lambda_1^{(n)} \leq \lambda_2^{(n)} \leq \ldots \leq \lambda_n^{(n)}.$$

The first Szegö limit theorem describes the collective behavior of the eigenvalues. It says in particular that under certain assumptions,

$$\frac{|\{j : \lambda_j^{(n)} \in (\alpha, \beta)\}|}{n} = \frac{|\{t \in \mathbf{T} : a(t) \in (\alpha, \beta)\}|}{2\pi} + o(1) \qquad (1)$$

as $n \to \infty$, where $|E|$ denotes the cardinality of E on the left and the Lebesgue measure of E on the right.

G.Szegö 1915, S.Parter 1986, F.Avram 1988, E.Tyrtyshnikov and N.Zamarashkin 1994-1998, P.Tilli 1998, 2002, A.Böttcher, S.Grudsky and E.Maximenko 2007.

Much attention has been paid to the extreme eigenvalues, that is, to the behavior of $\lambda_j^{(n)}$ as $n \to \infty$ and j or $n - j$ remain fixed. The pioneering work on this problem was done by Kac, Murdock, Szegö (1953), Widom (1958) and Parter (1961).
Recent work on and applications of extreme eigenvalues include the authors:

- S.Serra Capizzano and P.Tilli 1996-1999,
- C.Hurvich and Yi Lu 2005,
- A.Novoseltsev and I.Simonenko 2005,
- A.Böttcher, S.Grudsky and E.Maximenko 2008.

H.Widom (1958)
$a = \bar{a}, \quad g(\varphi) := a(e^{i\varphi}), \quad g(0) = 0, \quad g'(0) = 0, \quad g''(0) > 0$

$$\lambda_j^{(n)} = \frac{g''(0)}{2}\left(\frac{\pi j}{n+1}\right)^2\left(1 + \frac{w_0}{n+1}\right) + O\left(\frac{1}{n^4}\right), \quad j - \text{fixed}$$

The purpose of this paper is to explore the behavior of $\lambda_j^{(n)}$ inside the set of the eigenvalues. That is the asymptotics of $\lambda_j^{(n)}$ as $n \to \infty$ uniformly by parameter $d := \frac{\pi j}{n+1} \in (0, \pi)$.

1. Tridiagonal Toeplitz Matrices

$$a_1(t) = a_{-1}t^{-1} + a_0 + a_1 t$$

$$\lambda_j^{(n)} = a_0 + 2\sqrt{a_1 a_{-1}} \quad \cos\frac{\pi j}{n+1}$$

2.

$$a_2(t) = \frac{1}{a_1(t)}$$

Throughout here we assume the following. The function a is a Laurent polynomial

$$a(t) = \sum_{k=-r}^{r} a_k t^k \quad (t = e^{ix} \in \mathbf{T})$$

with $r \geq 1$, $a_r \neq 0$, and $\overline{a_k} = a_{-k}$ for all k. The last condition means that a is real-valued on $\mathbf{T}$. It may be assumed without loss of generality that $a(\mathbf{T}) = [0, M]$ with $M > 0$ and that $a(1) = 0$ and $a(e^{i\varphi_0}) = M$ for some $\varphi_0 \in (0, 2\pi)$. We require that the function $g(x) := a(e^{ix})$ is strictly increasing on $(0, \varphi_0)$ and strictly decreasing on $(\varphi_0, 2\pi)$ and that the second derivatives of g at $x = 0$ and $x = \varphi_0$ are nonzero. For each $\lambda \in [0, M]$, there exist exactly one $\varphi_1(\lambda) \in [0, \varphi_0]$ and exactly one $\varphi_2(\lambda) \in [\varphi_0 - 2\pi, 0]$ such that

$$g(\varphi_1(\lambda)) = g(\varphi_2(\lambda)) = \lambda;$$

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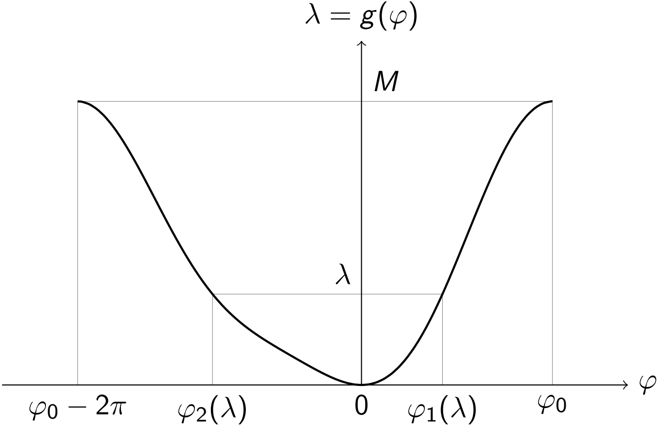

We put

$$\varphi(\lambda) = \frac{\varphi_1(\lambda) - \varphi_2(\lambda)}{2}.$$

Clearly, $\varphi(0) = 0$, $\varphi(M) = \pi$, φ is a continuous and strictly increasing map of $[0, M]$ onto $[0, \pi]$.

For $\lambda \in \mathbf{C}$, we write $a - \lambda$ in the form

$$\begin{aligned} a(t) - \lambda &= t^{-r}(a_r t^{2r} + \ldots + (a_0 - \lambda)t^r + \ldots + a_{-r}) \\ &= a_r t^{-r} \prod_{k=1}^{2r} (t - z_k(\lambda)) \qquad (2) \end{aligned}$$

with complex numbers $z_k(\lambda)$. We may label the zeros $z_1(\lambda), \ldots, z_{2r}(\lambda)$ so that each z_k is a continuous function of $\lambda \in \mathbf{C}$. Now take $\lambda \in [0, M]$. Then $a - \lambda$ has exactly the two zeros $e^{i\varphi_1(\lambda)}$ and $e^{i\varphi_2(\lambda)}$ on $\mathbf{T}$. We put

$$z_r(\lambda) = e^{i\varphi_1(\lambda)}, \quad z_{r+1}(\lambda) = e^{i\varphi_2(\lambda)}.$$

For $t \in \mathbf{T}$ we have (2) on the one hand, and since $a(t) - \lambda$ is real, we get

$$\begin{aligned} a(t) - \lambda &= \overline{a(t) - \lambda} = \overline{a_r}\, t^r \prod_{k=1}^{2r} \left(\frac{1}{t} - \overline{z}_k(\lambda) \right) \\ &= \overline{a_r} \left(\prod_{k=1}^{2r} \overline{z}_k(\lambda) \right) t^{-r} \prod_{k=1}^{2r} \left(t - \frac{1}{\overline{z}_k(\lambda)} \right) \end{aligned} \tag{3}$$

Comparing (2) and (3) we see that the zeros in $\mathbf{C} \setminus \mathbf{T}$ may be relabeled so that they appear in pairs $z_k(\lambda), 1/\overline{z}_k(\lambda)$ with $|z_k(\lambda)| > 1$. Put $u_k(\lambda) = z_k(\lambda)$ for $1 \le k \le r-1$. We relabel $z_{r+2}(\lambda), \dots, z_{2r}(\lambda)$ to get $z_{2r-k}(\lambda) = 1/\overline{u}_k(\lambda)$ for $1 \le k \le r-1$. In summary, for $\lambda \in [0, M]$ we have

$$\begin{aligned} \mathcal{Z} &:= \{z_1(\lambda), \dots, z_{r-1}(\lambda), e^{i\varphi_1(\lambda)}, e^{i\varphi_2(\lambda)}, z_{r+2}(\lambda), \dots, z_{2r}(\lambda)\} \\ &= \{u_1(\lambda), \dots, u_{r-1}(\lambda), e^{i\varphi_1(\lambda)}, e^{i\varphi_2(\lambda)}, 1/\overline{u}_{r-1}(\lambda), \dots, 1/\overline{u}_1(\lambda)\}. \end{aligned} \tag{4}$$

Put

$$h_\lambda(z) = \prod_{k=1}^{r-1} \left(1 - \frac{z}{u_k(\lambda)}\right), \quad \sigma(\lambda) = \frac{\varphi_1(\lambda) + \varphi_2(\lambda)}{2},$$

$$d_0(\lambda) = (-1)^r a_r e^{i\sigma(\lambda)} \prod_{k=1}^{r-1} u_k(\lambda). \tag{5}$$

For $t \in \mathbf{T}$ we then may write

$$a(t) - \lambda = d_0(\lambda) e^{i\varphi(\lambda)} \left(1 - \frac{t}{e^{i\varphi_1(\lambda)}}\right) \left(1 - \frac{e^{i\varphi_2(\lambda)}}{t}\right) h_\lambda(t) \overline{h_\lambda(t)}.$$

Widom's formula

H.Widom proved that if $\lambda \in \mathbf{C}$ and the points $z_1(\lambda), \ldots, z_{2r}(\lambda)$ are pairwise distinct, then the determinant of $T_n(a - \lambda)$ is

$$\det T_n(a - \lambda) = \sum_{J \subset \mathcal{Z}, |J| = r} C_J W_J^n \tag{6}$$

where the sum is over all subsets J of cardinality r of the set $\mathcal{Z}$ given by (4) and, with $\overline{J} := \mathcal{Z} \setminus J$,

$$C_J = \prod_{z \in J} z^r \prod_{z \in J, w \in \overline{J}} \frac{1}{z - w}, \quad W_J = (-1)^r a_r \prod_{z \in J} z.$$

Lemma (1)

Let $\lambda \in (0, M)$ *and put*

$$J_1 = \{u_1, \ldots, u_{r-1}, e^{i\varphi_1}\}, \quad J_2 = \{u_1, \ldots, u_{r-1}, e^{i\varphi_2}\}.$$

Then

$$W_{J_1} = d_0 e^{i\varphi}, \quad C_{J_1} = \frac{d_1 e^{i(\varphi+\theta)}}{2i \sin\varphi},$$

$$W_{J_2} = d_0 e^{-i\varphi}, \quad C_{J_2} = -\frac{d_1 e^{-i(\varphi+\theta)}}{2i \sin\varphi}.$$

Where $d_0 := d_0(\lambda) = (-1)^r a_r e^{i\sigma(\lambda)} \prod_{k=1}^{r-1} u_k(\lambda)$; $\qquad \varphi(\lambda) := \varphi = \frac{\varphi_1 - \varphi_2}{2}$.

$$d := d_1(\lambda) = \frac{1}{|h_\lambda(e^{i\varphi_1(\lambda)})h_\lambda(e^{i\varphi_2(\lambda)})|} \prod_{k,s=1}^{r-1} \left(1 - \frac{1}{u_k(\lambda)\overline{u}_s(\lambda)}\right)^{-1} \qquad (7)$$

$$\Theta(\lambda) := \frac{h_\lambda(e^{i\varphi_1(\lambda)})}{h_\lambda(e^{i\varphi_2(\lambda)})} = \prod_{k=1}^{r-1} \frac{1 - e^{i\varphi_1(\lambda)}/u_k(\lambda)}{1 - e^{i\varphi_2(\lambda)}/u_k(\lambda)}.$$

$$\theta := \theta(\lambda) := \arg \Theta(\lambda).$$

Theorem (A)

For every $\lambda \in (0, M)$ and every $\delta < \delta_0$,

$$\det T_n(a - \lambda) = \frac{d_1(\lambda) d_0^n(\lambda)}{\sin \varphi(\lambda)} \left[\sin\left((n+1)\varphi(\lambda) + \theta(\lambda)\right) + O(e^{-\delta n})\right].$$

Lemma (2)

There is a natural number $n_0 = n_0(a)$ *such that if* $n \geq n_0$*, then the function*

$$f_n : [0, M] \to [0, (n+1)\pi], \quad f_n(\lambda) = (n+1)\varphi(\lambda) + \theta(\lambda)$$

is bijective and increasing.

Main result.

Theorem (1)

If n is sufficiently large, then the function

$$[0, M] \to [0, (n+1)\pi], \quad \lambda \mapsto (n+1)\varphi(\lambda) + \theta(\lambda)$$

is bijective and increasing. For $1 \le j \le n$*, the eigenvalues* $\lambda_j^{(n)}$ *satisfy*

$$(n+1)\varphi(\lambda_j^{(n)}) + \theta(\lambda_j^{(n)}) = \pi j + O(e^{-\delta n}),$$

and if $\lambda_{j,*}^{(n)} \in (0, M)$ *is the uniquely determined solution of the equation*

$$(n+1)\varphi(\lambda_{j,*}^{(n)}) + \theta(\lambda_{j,*}^{(n)}) = \pi j,$$

then $|\lambda_j^{(n)} - \lambda_{j,*}^{(n)}| = O(e^{-\delta n})$*.*

Iteration procedure.

Here is an iteration procedure for approximating the numbers $\lambda_{j,*}^{(n)}$ and thus the eigenvalues $\lambda_j^{(n)}$. We know that $\varphi : [0, M] \to [0, \pi]$ is bijective and increasing. Let $\psi : [0, \pi] \to [0, M]$ be the inverse function. The equation

$$(n+1)\varphi(\lambda) + \theta(\lambda) = \pi j$$

is equivalent to the equation

$$\lambda = \psi\left(\frac{\pi j - \theta(\lambda)}{n+1}\right).$$

We define $\lambda_{j,0}^{(n)}, \lambda_{j,1}^{(n)}, \lambda_{j,2}^{(n)}, \ldots$ iteratively by

$$\lambda_{j,0}^{(n)} = \psi\left(\frac{\pi j}{n+1}\right), \quad \lambda_{j,k+1}^{(n)} = \psi\left(\frac{\pi j - \theta(\lambda_{j,k}^{(n)})}{n+1}\right) \text{ for } k = 0, 1, 2, \ldots.$$

Put $$\gamma = \sup_{\lambda \in (0,M)} \left|\frac{\theta'(\lambda)}{\varphi'(\lambda)}\right|.$$

Theorem (2)

There is a constant γ_0 depending only on a such that if n is sufficiently large, then

$$|\lambda_{j,k}^{(n)} - \lambda_{j,*}^{(n)}| \leq \gamma_0 \left(\frac{\gamma}{n+1}\right)^k \frac{1}{n+1} \frac{|\theta(\lambda_{j,0}^{(n)})|}{\varphi'(\lambda_{j,0}^{(n)})}$$

for all $1 \leq j \leq n$ and all $k \geq 0$.

Asymptotics of the eigenvalues.

Theorem (3)

We have

$$\lambda_j^{(n)} = \psi(d) - \frac{\psi\prime(d)\theta(\psi(d))}{n+1} + O\left(\frac{(\theta(\psi(d)))^2}{n^2}\right) + O\left(\frac{\psi\prime(d)\theta(\psi(d))}{n^2}\right).$$

Where $d = \frac{\pi j}{n+1}$ and $O(.)$ means that

$$O\left(\frac{(\theta(\psi(d)))^2 + \psi\prime(d)\theta(\psi(d))}{n^2}\right) \leq \mathrm{const}\,\frac{(\theta(\psi(d)))^2 + \psi\prime(d)\theta(\psi(d))}{n^2}.$$

Where "const" does not depend of n and $d \in (0, \pi)$. In particular

$$\lambda_j^{(n)} = \psi(d) - \frac{\psi\prime(d)\theta(\psi(d))}{n+1} + O\left(\frac{1}{n^2}\right), \qquad (8)$$

uniformly in d from compact subsets of $(0, \pi)$.

This is asymptotics for inner eigenvalues!

Asymptotic for extreme eigenvalues.

Theorem (4)

If $n \to \infty$ *and* $j/n \to 0$, *then*

$$\lambda_j^{(n)} = \sum_{k=0}^{3} (-1)^k \frac{\psi^{(k)}(d)}{k!} \left(\frac{\theta(\psi(d))}{n+1} \right)^k + O\left(\frac{1}{n^4}\right) \quad (9)$$

$$= \frac{g''(0)}{2} \left(\frac{\pi j}{n+1} \right)^2 \left(1 + \frac{w_0}{n+1} \right) + O\left(\frac{j^4}{n^4}\right) \quad (10)$$

$$= \frac{g''(0)}{2} \left(\frac{\pi j}{n+1} \right)^2 + O\left(\frac{j^3}{n^3}\right), \quad (11)$$

$$w_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{g'(x)}{g(x)} - \cot\frac{x}{2} - \frac{g'''(0)}{3g''(0)} \right) \cot\frac{x}{2}\, dx. \quad (12)$$

(10) coincides with Widom's formula. But (10) holds if $d = \frac{\pi j}{n+1} \ll 1$, while Widom's formula holds for j is fixed.

Even case.

Let be $g(-\varphi) = g(\varphi)$, $(g(\varphi) = a(e^{i\varphi}))$, then $g(\pi) = M$, $\varphi_1(\lambda) = -\varphi_2(\lambda) \in [0, \pi]$, $\varphi(\lambda) = \frac{\varphi_1(\lambda) - \varphi_2(\lambda)}{2} = \varphi_1(\lambda)$ and function $\psi(x) := \varphi^{-1}(x) = g(x)$.
This the main formula has the form

$$\lambda_j^{(n)} = g(d) - \frac{g\prime(d)\theta(g(d))}{n+1} + O\left(\frac{1}{n^2}\right).$$

Remark

Starting with $\lambda_{j,2}^{(n)}, \lambda_{j,3}^{(n)}, \ldots$ instead of $\lambda_{j,1}^{(n)}$ one can get as many terms of the expansions (8) or (9) as desired.

Examples.

We consider $T_n(a)$, denote by $\lambda_j^{(n)}$ the jth eigenvalue, by $\lambda_{j,*}^{(n)}$ the approximation to $\lambda_j^{(n)}$ given by Theorem (1), and by $\lambda_{j,k}^{(n)}$ the kth approximation to $\lambda_j^{(n)}$ delivered by the iteration procedure. We put

$$\Delta_*^{(n)} = \max_{1 \le j \le n} |\lambda_j^{(n)} - \lambda_{j,*}^{(n)}|, \quad \Delta_k^{(n)} = \max_{1 \le j \le n} |\lambda_j^{(n)} - \lambda_{j,k}^{(n)}|.$$

We let w_0 be the constant (12), denote by

$$\lambda_{j,W}^{(n)} = \frac{g''(0)}{2} \left(\frac{\pi j}{n+1} \right)^2 \left(1 + \frac{w_0}{n+1} \right)$$

Widom's approximation for the jth extreme eigenvalue given by (10), and put

$$\Delta_{j,W}^{(n)} = \frac{(n+1)^4}{\pi^4 j^4} |\lambda_j^{(n)} - \lambda_{j,W}^{(n)}|.$$

Example (1)

(*A symmetric pentadiagonal matrix*) Let $a(t) = 8 - 5t - 5t^{-1} + t^2 + t^{-2}$. In that case

$$g(x) = 8 - 10\cos x + 2\cos 2x = 4\sin^2\frac{x}{2} + 16\sin^4\frac{x}{2},$$

$a(\mathbf{T}) = [0, 20]$, and for $\lambda \in [0, 20]$, the roots of $a(z) - \lambda$ are $e^{-i\varphi(\lambda)}$, $e^{i\varphi(\lambda)}$, $u(\lambda)$, $1/u(\lambda)$ with

$$\varphi(\lambda) = \arccos\frac{5 - \sqrt{1 + 4\lambda}}{4} = 2\arcsin\frac{\sqrt{\sqrt{1 + 4\lambda} - 1}}{2\sqrt{2}},$$

$$u(\lambda) = \frac{5 + \sqrt{1 + 4\lambda}}{4} + \frac{\sqrt{5 + 2\lambda + 5\sqrt{1 + 4\lambda}}}{2\sqrt{2}}$$

and we have

$$g''(0) = 2, \quad w_0 = \frac{4}{u(0) - 1} = 2\sqrt{5} - 2.$$

Example (1)

The errors $\Delta_*^{(n)}$ *are*

	$n=10$	$n=20$	$n=50$	$n=100$	$n=150$
$\Delta_*^{(n)}$	$5.4\cdot10^{-7}$	$1.1\cdot10^{-11}$	$5.2\cdot10^{-25}$	$1.7\cdot10^{-46}$	$9.6\cdot10^{-68}$

and for $\Delta_k^{(n)}$ *and* $\Delta_{j,W}^{(n)}$ *we have*

	$n=10$	$n=100$	$n=1000$	$n=10000$
$\Delta_1^{(n)}$	$9.0\cdot10^{-2}$	$1.1\cdot10^{-4}$	$1.1\cdot10^{-6}$	$1.1\cdot10^{-8}$
$\Delta_2^{(n)}$	$2.2\cdot10^{-4}$	$2.8\cdot10^{-7}$	$2.9\cdot10^{-10}$	$2.9\cdot10^{-13}$
$\Delta_3^{(n)}$	$1.1\cdot10^{-5}$	$1.5\cdot10^{-9}$	$1.5\cdot10^{-13}$	$1.5\cdot10^{-17}$

Example (1)

	$n = 10$	$n = 100$	$n = 1000$	$n = 10000$	$n = 100000$
$\Delta_{1,W}^{(n)}$	1.462	1.400	1.383	1.381	1.381
$\Delta_{2,W}^{(n)}$	0.997	1.046	1.034	1.033	1.033
$\Delta_{3,W}^{(n)}$	0.840	0.979	0.970	0.968	0.968

Example (2)

(A Hermitian heptadiagonal matrix)

$$a(t) = 24 + (-12 - 3i)t + (-12 + 3i)t^{-1} + it^3 - it^{-3},$$
$$g(x) = 48\sin^2\frac{x}{2} + 8\sin^3 x.$$

	$n = 10$	$n = 20$	$n = 50$	$n = 100$	$n = 150$
$\Delta_*^{(n)}$	$6.6 \cdot 10^{-6}$	$1.2 \cdot 10^{-10}$	$7.6 \cdot 10^{-24}$	$1.4 \cdot 10^{-45}$	$3.3 \cdot 10^{-67}$

	$n = 10$	$n = 100$	$n = 1000$	$n = 10000$
$\Delta_1^{(n)}$	$1.0 \cdot 10^{-2}$	$1.4 \cdot 10^{-4}$	$1.5 \cdot 10^{-6}$	$1.5 \cdot 10^{-8}$
$\Delta_2^{(n)}$	$3.2 \cdot 10^{-4}$	$5.8 \cdot 10^{-7}$	$5.9 \cdot 10^{-10}$	$5.9 \cdot 10^{-13}$
$\Delta_3^{(n)}$	$1.4 \cdot 10^{-5}$	$2.4 \cdot 10^{-9}$	$2.5 \cdot 10^{-13}$	$2.6 \cdot 10^{-17}$

Example (2)

	$n = 10$	$n = 100$	$n = 1000$	$n = 10000$	$n = 100000$
$\Delta_{1,W}^{(n)}$	5.149	7.344	7.565	7.587	7.589
$\Delta_{2,W}^{(n)}$	4.106	7.386	7.623	7.645	7.647
$\Delta_{3,W}^{(n)}$	2.606	7.370	7.633	7.656	7.658

Formulas for the eigenvectors (symmetric case)

Introduce the vectors $y_k^{(n)}$ with the following coordinates:

$$y_{k,m}^{(n)} := \sin\left(m\varphi(\lambda) + \frac{\theta(\lambda)}{2}\right) - \sum_{j=1}^{r-1} Q_j(\lambda)\left(\frac{1}{u_j(\lambda)^m} + \frac{(-1)^{k+1}}{u_j(\lambda)^{n+1-m}}\right),$$

$$\text{where} \quad Q_j(\lambda) = \frac{|h_\lambda(e^{i\varphi(\lambda)})| \sin\varphi(\lambda)}{(u_j(\lambda) - e^{i\varphi(\lambda)})(u_j(\lambda) - e^{i\varphi(\lambda)})h'_\lambda(u_j(\lambda))}, \quad \lambda = \lambda_k^{(n)}.$$

Let $w_k^{(n)}$ be the normalized vector $y_k^{(n)}$ and v_n^k be normalized eigenvector.

Theorem (5)

$$\varrho(v_k^{(n)}, w_k^{(n)}) \le Ce^{-n\delta},$$

where C and δ depend only on the symbol.

In the nonsymmetric case the formulas for $y_k^{(n)}$ are a little more complicated.

Numerical results

Given $T_n(a)$, determine the approximate eigenvalue $\lambda_{j,*}^{(n)}$ from the equation

$$(n+1)\varphi(\lambda_{j,*}^{(n)}) + \theta(\lambda_{j,*}^{(n)}) = \pi j.$$

Put

$$w_{j,*}^{(n)} = \frac{w_j^{(n)}(\lambda_{j,*}^{(n)})}{\|w_j^{(n)}(\lambda_{j,*}^{(n)})\|_2}.$$

We define the distance between the normalized eigenvector $v_j^{(n)}$ and the normalized vector $w_{j,*}^{(n)}$ by

$$\varrho(v_j^{(n)}, w_{j,*}^{(n)}) := \min_{\tau \in \mathbf{T}} \|\tau v_j^{(n)} - w_{j,*}^{(n)}\|_2 = \sqrt{2 - 2\langle v_j^{(n)}, w_{j,*}^{(n)} \rangle}$$

and put

$$\Delta_*^{(n)} = \max_{1\le j\le n} |\lambda_j^{(n)} - \lambda_{j,*}^{(n)}|,$$

$$\Delta_{v,w}^{(n)} = \max_{1\le j\le n} \varrho(v_j^{(n)}, w_{j,*}^{(n)}),$$

$$\Delta_r^{(n)} = \max_{1\le j\le n} \| T_n(a) w_{j,*}^{(n)}) - \lambda_{j,*}^{(n)} w_{j,*}^{(n)} \|_2.$$

The tables following below show these errors for three concrete choices of the generating function a.

For $a(t) = 8 - 5t - 5t^{-1} + t^2 + t^{-2}$ we have

	$n = 10$	$n = 20$	$n = 50$	$n = 100$	$n = 150$
$\Delta_*^{(n)}$	$5.4 \cdot 10^{-7}$	$1.1 \cdot 10^{-11}$	$5.2 \cdot 10^{-25}$	$1.7 \cdot 10^{-46}$	$9.6 \cdot 10^{-68}$
$\Delta_{v,w}^{(n)}$	$2.0 \cdot 10^{-6}$	$1.1 \cdot 10^{-10}$	$2.0 \cdot 10^{-23}$	$1.9 \cdot 10^{-44}$	$2.0 \cdot 10^{-65}$
$\Delta_r^{(n)}$	$8.0 \cdot 10^{-6}$	$2.7 \cdot 10^{-10}$	$3.4 \cdot 10^{-23}$	$2.2 \cdot 10^{-44}$	$1.9 \cdot 10^{-65}$

If $a(t) = 8 + (-4 - 2i)t + (-4 - 2i)t^{-1} + it - it^{-1}$ then

	$n = 10$	$n = 20$	$n = 50$	$n = 100$	$n = 150$
$\Delta_*^{(n)}$	$3.8 \cdot 10^{-8}$	$2.8 \cdot 10^{-13}$	$2.9 \cdot 10^{-30}$	$5.9 \cdot 10^{-58}$	$1.6 \cdot 10^{-85}$
$\Delta_{v,w}^{(n)}$	$1.8 \cdot 10^{-7}$	$4.7 \cdot 10^{-13}$	$2.0 \cdot 10^{-29}$	$7.0 \cdot 10^{-57}$	$2.4 \cdot 10^{-84}$
$\Delta_r^{(n)}$	$5.4 \cdot 10^{-7}$	$1.3 \cdot 10^{-12}$	$2.7 \cdot 10^{-29}$	$6.7 \cdot 10^{-57}$	$1.9 \cdot 10^{-84}$

In the case where $a(t) = 24 + (-12 - 3i)t + (-12 + 3i)t^{-1} + it^3 - it^{-3}$ we get

	$n = 10$	$n = 20$	$n = 50$	$n = 100$	$n = 150$
$\Delta_*^{(n)}$	$6.6 \cdot 10^{-6}$	$1.2 \cdot 10^{-10}$	$7.6 \cdot 10^{-24}$	$1.4 \cdot 10^{-45}$	$3.3 \cdot 10^{-67}$
$\Delta_{v,w}^{(n)}$	$1.9 \cdot 10^{-6}$	$1.3 \cdot 10^{-10}$	$2.0 \cdot 10^{-23}$	$7.2 \cdot 10^{-45}$	$2.8 \cdot 10^{-66}$
$\Delta_r^{(n)}$	$2.5 \cdot 10^{-5}$	$8.6 \cdot 10^{-10}$	$7.3 \cdot 10^{-23}$	$1.9 \cdot 10^{-44}$	$5.9 \cdot 10^{-66}$