Composition of Muckenhoupt weights with inner function and the theory of Toeplitz operators with oscillating symbols.

Sergei Grudsky

(Cancun, April 2008)

This report is founded on joint works with

A.Böttcher and E. Shargorodsky.

1. *Böttcher A., Grudsky S.* On the composition of Muckenhoupt weights and inner functions. J. London Mathematical Society, (2) 58, N 1, 1998, 172–184.

2. *Grudsky S., Shargorodsky E.* Spectra of Toeplitz operators and compositions of Muckenhoupt weights with Blaschke products. Integral Equations and Operator Theory. 2008

Sufficient conditions on Blaschke products.

$$B(t) = \prod_{k=1}^{\infty} \tfrac{r_k - t}{1 - r_k t} \ , \qquad r_k \in (0,1) \ ,$$

and $\displaystyle\sum_{k=1}^{\infty} (1 - r_k) < \infty$

Theorem 4. Suppose
$r_1 \le r_2 \le \cdots \le r_n \le \cdots$ and

$\inf \frac{1-r_{k+1}}{1-r_k} > 0$

if $\omega \in A_p$, then $\omega \circ B \in A_p$

Theorem 5. Let $1 < p < \infty$, $a \in L^\infty(\mathbb{T})$, and let a Blaschke product B satisfy the conditions of Theorem 4. Then

$$T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$$

is invertible if and only if

$$T(a \circ B) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$$

is invertible.

Logaritmic case

Theorem 6. Suppose a real valued function η is continuous on $[-\pi, \pi] \setminus \{0\}$ and

$$\lim_{t \to 0 \pm 0} (\eta(t) \mp \pi \log |t|) = 0.$$

Then the function $e^{i\eta}$ admits the following representation

$$e^{i\eta(t)} = B\left(e^{it}\right) g\left(B\left(e^{it}\right)\right) d\left(e^{it}\right), \quad t \in [-\pi, \pi],$$

where $g, d \in C(\mathbb{T})$, the winding number of g is 0, and B is the infinite Blaschke product with the zeros

$$r_k = \frac{2 - \exp(-k + 1/2)}{2 + \exp(-k + 1/2)}.$$

Theorem 7. Suppose a function $\eta(t)$ satisfies the conditions of Theorem 6. Then the operator $T(e^{i\eta(t)})$ is left invertible on the space $H^p(\mathbb{T})$, for $1 < p < \infty$.

Aplication to problem about spectra of Toeplitz operators with symbols having more then two limiting values.

Definition 1. Let $\mathbb{T} = \{\zeta \in \mathbb{C} : \ |\zeta| = 1\}$ be the unit circle. A number $c \in \mathbb{C}$ is called a *(left, right) cluster value* of a measurable function $a : \mathbb{T} \to \mathbb{C}$ at a point $\zeta \in \mathbb{T}$ if $1/(a-c) \notin L^\infty(W)$ for every neighbourhood (left semi-neighbourhood, right semi-neighbourhood) $W \subset \mathbb{T}$ of ζ.

We denote the set of all left (right) cluster values of a at ζ by $a(\zeta - 0)$ (by $a(\zeta + 0)$), and use also the following notation

$$a(\zeta) = a(\zeta - 0) \cup a(\zeta + 0),$$

$$a(\mathbb{T}) = \cup_{\zeta \in \mathbb{T}} a(\zeta).$$

Let $K \subset \mathbb{C}$ be an arbitrary compact set and $\lambda \in \mathbb{C} \setminus K$. Then the set

$$\sigma(K;\lambda) = \left\{ \frac{w-\lambda}{|w-\lambda|} \;\middle|\; w \in K \right\} \subseteq \mathbb{T}$$

is compact as a continuous image of a compact set. Hence the set $\Delta_\lambda(K) := \mathbb{T} \setminus \sigma(K;\lambda)$ is open in $\mathbb{T}$. So, $\Delta_\lambda(K)$ is the union of an at most countable family of open arcs.

Definition 2. We call an open arc of $\mathbb{T}$ p–large if its length is greater than or equal to

$$\tfrac{2\pi}{\max\{p,q\}}, \quad \text{where } q = \tfrac{p}{p-1}, \quad 1 < p < \infty.$$

Theorem 8. (E.Shargorodsky)
Let $1 < p < \infty$, $a \in L^\infty(\mathbb{T})$, $\lambda \in \mathbb{C} \setminus a(\mathbb{T})$ and suppose that, for some $\zeta \in \mathbb{T}$,
(i) $\Delta_\lambda(a(\zeta - 0))$ (or $\Delta_\lambda(a(\zeta + 0))$) contains at least two p–large arcs,
(ii) $\Delta_\lambda(a(\zeta + 0))$ (or $\Delta_\lambda(a(\zeta - 0))$ respectively) contains at least one p–large arc.
Then λ belongs to the essential spectrum of $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$.

E.Shargorodsky (1994) was shown that condition (i) is optimal in the following sense: for any compact $K \subset \mathbb{C}$ and $\lambda \in \mathbb{C} \setminus K$ such that $\Delta_\lambda(K)$ contains at most one p–large arc there exists $a \in L^\infty(\mathbb{T})$ such that $a(-1 \pm 0) = a(\mathbb{T}) = K$ and $T(a) - \lambda I : H^r(\mathbb{T}) \to H^r(\mathbb{T})$ is invertible for any $r \in [\min\{p, q\}, \max\{p, q\}]$. A question that has been open is whether or not condition (ii) can be dropped, i.e. whether condition (i) alone is sufficient for λ to belong to the essential spectrum of $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$. The following result gives a negative answer to this question.

Theorem 9. There exists $a \in L^\infty(\mathbb{T})$ such that $a(1-0) = \{\pm 1\}$, $|a| \equiv 1$,

$$T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$$

is invertible for any $p \in (1,2)$, and

$$T(1/a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$$

is invertible for any $p \in (2,+\infty)$.

Proof of Theorem 9.

Let $a_0 \in L^\infty(\mathbb{T})$ be defined by

$$a_0(e^{i\tau}) = \exp\left(i\,\frac{\tau}{2}\right)\,, \quad \tau \in (0,2\pi).$$

Then a_0 is continuous on $\mathbb{T} \setminus \{1\}$,
$a_0(1 \pm 0) = \pm 1$, $T(a_0) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is invertible for any $p \in (1,2)$,
and $T(1/a_0) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is invertible for any $p \in (2,+\infty)$.

Let $h_0(z) = \exp\big(-ic\log(i\frac{1-z}{z}\big)$. Then

$$h_0(e^{it}) = |h_0(e^{it})|\, e^{i\varphi(t)}\,, \quad t \in [-\pi,\pi],$$

where

$$\varphi(t) = -\frac{\pi}{2}\,\log\left|2\sin\frac{t}{2}\right|$$

Let f be a 2π-periodic function such that

$$f \in C^\infty([-\pi, \pi] \setminus \{0\}),$$
$$f(t) = \varphi(t) \text{ if } -\pi/2 \le t < 0,$$

and

$$f(t) = -f(-t) \text{ if } 0 < t \le \pi/2.$$

Then (Theorem 6)

$$e^{2if(t)} = B\left(e^{it}\right) g\left(B\left(e^{it}\right)\right) d\left(e^{it}\right), \qquad (1)$$

$t \in [-\pi, \pi]$, where $g, d \in C(\mathbb{T})$, the index of g is 0, and B is the infinite Blaschke product with the zeros

$$r_k = \frac{2 - \exp(-k + 1/2)}{2 + \exp(-k + 1/2)}$$

Since the index of g is 0, there exists $g_0 \in C(\mathbb{T})$ such that $g_0^2 = g$. Let $d_0 \in C(\mathbb{T})$ be such that $d_0^2(e^{it}) = d(e^{it})$ for $t \in [-\pi/2, \pi/2]$, $d_0(e^{it}) \neq 0$ for $t \in [-\pi, \pi]$ and the index of d_0 is 0.

Consider the function $a \in L^\infty(\mathbb{T})$ defined by

$$a(e^{it}) = a_0\left(B\left(e^{it}\right)\right) \left(\frac{g_0\left(B\left(e^{it}\right)\right) d_0\left(e^{it}\right) |h_0(e^{it})|}{h_0(e^{it})} \right). \qquad (2)$$

It follows from (1) and from the definition of the function f that $a^2(e^{it}) = 1$ if $-\pi/2 \leq t < 0$. It is clear that the second factor in the right-hand side of (2) is continuous on
$\{e^{it} | \; -\pi/2 \leq t < 0\}$, whereas the first one has infinitely many discontinuities in any left semi-neighbourhood of 1. Hence a takes values 1 and -1 in any left semi-neighbourhood of 1. So, $a(1-0) = \{\pm 1\}$.

The operator $T(a^{\pm 1}) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is invertible if and only if
$T(a_0^{\pm 1} \circ B) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is invertible. The latter operator is indeed invertible because $T(a_0^{\pm 1}) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is invertible and B satisfies of Theorem 4.