On a quaternionic Maxwell equation for the time-dependent electromagnetic field in a chiral medium

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February 5, 2008

Abstract

Maxwell's equations for the time-dependent electromagnetic field in a homogeneous chiral medium are reduced to a single quaternionic equation. Its fundamental solution satisfying the causality principle is obtained which allows us to solve the time-dependent chiral Maxwell system with sources.

1 Introduction

We consider Maxwell's equations for the time-dependent electromagnetic field in a homogeneous chiral medium and show their equivalence to a single quaternionic equation. This result generalizes the well known (see [13], [6], [10]) quaternionic reformulation of the Maxwell equations for non-chiral media. Nevertheless the new quaternionic differential operator is essentially different from the quaternionic operator corresponding to the non-chiral case. We obtain a fundamental solution of the new operator in explicit form satisfying the causality principle. Its convolution with a quaternionic function representing sources of the electromagnetic field gives us a solution of the inhomogeneous Maxwell system in a whole space.

2 Maxwell's equations for chiral media

Consider time-dependent Maxwell's equations

$$\operatorname{rot} \overrightarrow{E}(t,x) = -\partial_t \overrightarrow{B}(t,x), \tag{1}$$

$$\operatorname{rot} \overrightarrow{H}(t,x) = \partial_t \overrightarrow{D}(t,x) + \overrightarrow{j}(t,x), \tag{2}$$

$$\operatorname{div} \overrightarrow{E}(t,x) = \frac{\rho(t,x)}{\varepsilon}, \qquad \operatorname{div} \overrightarrow{H}(t,x) = 0 \tag{3}$$

with the Drude-Born-Fedorov constitutive relations corresponding to the chiral media [2], [11], [12]

$$\overrightarrow{B}(t,x) = \mu(\overrightarrow{H}(t,x) + \beta \operatorname{rot} \overrightarrow{H}(t,x)), \tag{4}$$

$$\overrightarrow{D}(t,x) = \varepsilon(\overrightarrow{E}(t,x) + \beta \operatorname{rot} \overrightarrow{E}(t,x)), \tag{5}$$

where β is the chirality measure of the medium. β, ε, μ are real scalars assumed to be constants. Note that the charge density ρ and the current density $\overrightarrow{j}$ are related by the continuity equation $\partial_t \rho + \operatorname{div} \overrightarrow{j} = 0$.

Incorporating the constitutive relations (4), (5) into the system (1)-(3) we arrive at the main object of our study, the time-dependent Maxwell system for a homogeneous chiral medium

$$\operatorname{rot} \overrightarrow{H}(t,x) = \varepsilon(\partial_t \overrightarrow{E}(t,x) + \beta \partial_t \operatorname{rot} \overrightarrow{E}(t,x)) + \overrightarrow{j}(t,x), \tag{6}$$

$$\operatorname{rot} \overrightarrow{E}(t,x) = -\mu(\partial_t \overrightarrow{H}(t,x) + \beta \partial_t \operatorname{rot} \overrightarrow{H}(t,x)), \tag{7}$$

$$\operatorname{div} \overrightarrow{E}(t,x) = \frac{\rho(t,x)}{\varepsilon}, \qquad \operatorname{div} \overrightarrow{H}(t,x) = 0. \tag{8}$$

Application of rot to (6) and (7) allows us to separate the equations for $\overrightarrow{E}$ and $\overrightarrow{H}$ and to obtain in this way the wave equations for a chiral medium

$$\operatorname{rot}\operatorname{rot} \overrightarrow{E} + \varepsilon\mu\partial_t^2 \overrightarrow{E} + 2\beta\varepsilon\mu\partial_t^2 \operatorname{rot} \overrightarrow{E} + \beta^2\varepsilon\mu\partial_t^2 \operatorname{rot}\operatorname{rot} \overrightarrow{E} = -\mu\partial_t \overrightarrow{j} - \beta\mu\partial_t \operatorname{rot} \overrightarrow{j}, \tag{9}$$

$$\operatorname{rot}\operatorname{rot} \overrightarrow{H} + \varepsilon\mu\partial_t^2 \overrightarrow{H} + 2\beta\varepsilon\mu\partial_t^2 \operatorname{rot} \overrightarrow{H} + \beta^2\varepsilon\mu\partial_t^2 \operatorname{rot}\operatorname{rot} \overrightarrow{H} = \operatorname{rot} \overrightarrow{j}. \tag{10}$$

It should be noted that when $\beta = 0$, (9) and (10) reduce to the wave equations for non-chiral media but in general to the difference of the usual non-chiral wave equations their chiral generalizations represent equations of fourth order.

3 Some notations from quaternionic analysis

We will consider biquaternion-valued functions defined in some domain $\Omega \subset \mathbb{R}^3$. On the set of continuously differentiable such functions the well known Moisil-Teodoresco operator is defined by the expression $D = i_1\frac{\partial}{\partial x_1} + i_2\frac{\partial}{\partial x_2} + i_3\frac{\partial}{\partial x_3}$ (see, e.g., [5]), where i_k, $k = 1,2,3$ are basic quaternionic imaginary units. Denote $D_\alpha = D + \alpha$, where $\alpha \in \mathbb{C}$ and $\operatorname{Im}\alpha \geq 0$. The fundamental solution for this operator is known [9] (see also [10]):

$$\mathcal{K}_\alpha(x) = -\operatorname{grad}\Theta_\alpha(x) + \alpha\Theta_\alpha(x) = (\alpha + \frac{x}{|x|^2} - i\alpha\frac{x}{|x|})\Theta_\alpha(x), \tag{11}$$

where i is the usual complex imaginary unit commuting with i_k, $x = \sum_{k=1}^3 x_k i_k$ and $\Theta_\alpha(x) = -\frac{e^{i\alpha|x|}}{4\pi|x|}$. Note that $\mathcal{K}_\alpha$ fulfills the following radiation condition at infinity uniformly in all directions

$$(1 + \frac{ix}{|x|}) \cdot \mathcal{K}_\alpha(x) = o(\frac{1}{|x|}), \qquad \text{when } |x| \to \infty \tag{12}$$

which is in agreement with the Silver-Müller radiation conditions [8].

4 Field equations in quaternionic form

In this section we rewrite the field equations from Section 2 in quaternionic form.

Let us introduce the following quaternionic operator

$$M = \beta\sqrt{\varepsilon\mu}\partial_t D + \sqrt{\varepsilon\mu}\partial_t - iD \tag{13}$$

and consider the purely vectorial biquaternionic function

$$\overrightarrow{V}(t,x) = \overrightarrow{E}(t,x) - i\sqrt{\frac{\mu}{\varepsilon}}\overrightarrow{H}(t,x). \tag{14}$$

Proposition 1 *The quaternionic equation*

$$M\overrightarrow{V}(t,x) = -\sqrt{\frac{\mu}{\varepsilon}}\overrightarrow{j}(t,x) - \beta\sqrt{\frac{\mu}{\varepsilon}}\partial_t\rho(t,x) + \frac{i\rho(t,x)}{\varepsilon} \tag{15}$$

is equivalent to the Maxwell system (6)-(8), the vectors $\overrightarrow{E}$ *and* $\overrightarrow{H}$ *are solutions of (6)-(8) if and only if the purely vectorial biquaternionic function* $\overrightarrow{V}$ *defined by (14) is a solution of (15).*

Proof. The scalar and the vector parts of (15) have the form

$$-\beta\sqrt{\varepsilon\mu}\partial_t \operatorname{div}\overrightarrow{E} + \sqrt{\frac{\mu}{\varepsilon}}\operatorname{div}\overrightarrow{H} + i(\operatorname{div}\overrightarrow{E} + \beta\mu\partial_t\operatorname{div}\overrightarrow{H}) = -\beta\sqrt{\frac{\mu}{\varepsilon}}\partial_t\rho + \frac{i\rho}{\varepsilon}, \tag{16}$$

$$\beta\sqrt{\varepsilon\mu}\partial_t \operatorname{rot}\overrightarrow{E} + \sqrt{\varepsilon\mu}\partial_t\overrightarrow{E} - \sqrt{\frac{\mu}{\varepsilon}}\operatorname{rot}\overrightarrow{H} - i(\operatorname{rot}\overrightarrow{E} + \beta\mu\partial_t\operatorname{rot}\overrightarrow{H} + \mu\partial_t\overrightarrow{H}) = -\sqrt{\frac{\mu}{\varepsilon}}\overrightarrow{j}. \tag{17}$$

The real part of (17) coincides with (6) and the imaginary part coincides with (7). Applying divergence to the equation (17) and using the continuity equation gives us

$$\partial_t \operatorname{div}\overrightarrow{H} = 0 \quad \text{and} \quad \partial_t \operatorname{div}\overrightarrow{E} = \frac{1}{\varepsilon}\partial_t\rho.$$

Taking into account these two equalities we obtain from (16) that the vectors $\overrightarrow{E}$ and $\overrightarrow{H}$ satisfy equations (8). ■

It should be noted that for $\beta = 0$ from (13) we obtain the operator which was studied in [7] with the aid of the factorization of the wave operator for non-chiral media

$$\varepsilon\mu\partial_t^2 - \Delta_x = (\sqrt{\varepsilon\mu}\partial_t + iD)(\sqrt{\varepsilon\mu}\partial_t - iD).$$

In the case under consideration we obtain a similar result. Let us denote by M^* the complex conjugate operator of M:

$$M^* = \beta\sqrt{\varepsilon\mu}\partial_t D + \sqrt{\varepsilon\mu}\partial_t + iD.$$

For simplicity we consider now a sourceless situation. In this case the equations (9) and (10) are homogeneous and can be represented as follows

$$MM^*\overrightarrow{U}(t,x) = 0,$$

where $\overrightarrow{U}$ stands for $\overrightarrow{E}$ or for $\overrightarrow{H}$.

5 Fundamental solution of the operator M

We will construct a fundamental solution of the operator M using the results of the previous section and well known facts from quaternionic analysis. Consider the equation

$$(\beta\sqrt{\varepsilon\mu}\partial_t D + \sqrt{\varepsilon\mu}\partial_t - iD)f(t,x) = \delta(t,x).$$

Applying the Fourier transform $\mathcal{F}$ with respect to the time-variable t we obtain

$$(\beta\sqrt{\varepsilon\mu}i\omega D + \sqrt{\varepsilon\mu}i\omega - iD)F(\omega,x) = \delta(x),$$

where $F(\omega,x) = \mathcal{F}\{f(t,x)\} = \int_{-\infty}^{\infty} f(t,x)e^{-i\omega t}dt$. The last equation can be rewritten as follows

$$(D+\alpha)(\beta\sqrt{\varepsilon\mu}\omega - 1)iF(\omega,x) = \delta(x),$$

where $\alpha = \frac{\sqrt{\varepsilon\mu}\omega}{\beta\sqrt{\varepsilon\mu}\omega - 1}$. The fundamental solution of D_α is given by (11), so we have

$$(\beta\sqrt{\varepsilon\mu}\omega - 1)iF(\omega,x) = (\alpha + \frac{x}{|x|^2} - i\alpha\frac{x}{|x|})\Theta_\alpha(x),$$

from where

$$F(\omega,x)=\left[\frac{i\sqrt{\varepsilon\mu}\omega}{(\beta\sqrt{\varepsilon\mu}\omega-1)^2}\left(1-\frac{ix}{|x|}\right)+\frac{ix}{|x|^2}\frac{1}{\beta\sqrt{\varepsilon\mu}\omega-1}\right]\frac{e^{i|x|\frac{\sqrt{\varepsilon\mu}\omega}{\beta\sqrt{\varepsilon\mu}\omega-1}}}{4\pi\,|x|}.$$

We write it in a more convenient form

$$F(\omega,x)=\left(\frac{1}{(\omega-a)^2}A\left(x\right)+\frac{1}{\omega-a}B\left(x\right)\right)E\left(x\right)e^{\frac{ic(x)}{\omega-a}},$$

where $a=\frac{1}{\beta\sqrt{\varepsilon\mu}}$, $c\left(x\right)=\frac{|x|}{\beta^2\sqrt{\varepsilon\mu}}$, $E\left(x\right)=\frac{e^{\frac{i|x|}{\beta}}}{4\pi|x|}$,

$$A\left(x\right)=\frac{i}{\beta^3\varepsilon\mu}\left(1-\frac{ix}{|x|}\right),\qquad B\left(x\right)=\frac{i}{\beta\sqrt{\varepsilon\mu}}\left(\frac{1}{\beta}\left(1-\frac{ix}{|x|}\right)+\frac{x}{|x|^2}\right).$$

In order to obtain the fundamental solution $f(t,x)$ we should apply the inverse Fourier transform to $F(\omega,x)$. Among different regularizations of the resulting integral we should choose the one leading to a fundamental solution satisfying the causality principle, that is vanishing for $t<0$. Such an election is done by introducing of a small parameter $y>0$ in the following way

$$f(t,x)=\lim_{y\to 0}\mathcal{F}^{-1}\left\{F(z,x)\right\} \tag{18}$$

where $z=\omega-iy$. This regularization is in agreement with the condition $\operatorname{Im}\alpha\geq 0$. We have

$$\mathcal{F}^{-1}\left\{F(z,x)\right\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{(\omega-a_y)^2}A\left(x\right)+\frac{1}{\omega-a_y}B\left(x\right)\right)E\left(x\right)e^{\frac{ic(x)}{\omega-a_y}}e^{i\omega t}d\omega \tag{19}$$

where $a_y=a+iy$. Expression (19) includes two integrals of the form

$$I_k=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\frac{ic}{\omega-a_y}}e^{i\omega t}}{(\omega-a_y)^k}d\omega,\quad k=1,2$$

where $c=c(x)$. We have

$$I_k=\frac{1}{2\pi}\sum_{j=0}^{\infty}\left(\frac{(ic)^j}{j!}\int_{-\infty}^{\infty}\frac{e^{i\omega t}d\omega}{(\omega-a_y)^{j+k}}\right). \tag{20}$$

Denote

$$I_{k,j}(t) = \int_{-\infty}^{\infty} \frac{e^{i\omega t} d\omega}{(\omega - a_y)^{j+k}}.$$

For $k = 1$ and $j = 0$ we obtain (see, e.g., [3, Sect. 8.7])

$$I_{1,0}(t) = 2\pi i H(t) e^{ita_y}$$

where H is the Heaviside function. For all other cases, that is for $k = 1$ and $j = \overline{1,\infty}$ and for $k = 2$ and $j = \overline{0,\infty}$ we have that $j + k \geq 2$ and the integrand in (20) has a pole at the point a_y of order $j + k$. Using a result from the residue theory [4, Sect. 4.3] we obtain

$$I_{k,j}(t) = 2\pi i \operatorname*{Res}_{a_y} \frac{e^{i\omega t}}{(\omega - a_y)^{j+k}} \quad \text{for } t \geq 0 \text{ and } j + k \geq 2.$$

Consider

$$\operatorname*{Res}_{a_y} \frac{e^{i\omega t}}{(\omega - a_y)^{j+k}} = \frac{1}{(j+k-1)!} \lim_{\omega \to a_y} \frac{\partial^{j+k-1}}{\partial \omega^{j+k-1}} e^{i\omega t} = \frac{(it)^{j+k-1} e^{ia_y t}}{(j+k-1)!} \quad \text{for } t \geq 0$$

and $j + k \geq 2$.

For $t < 0$ we have that $I_{k,j}(t)$ is equal to the sum of residues with respect to singularities in the lower half-plane $y < 0$ which is zero because the integrand is analytic there. Thus we obtain

$$I_{k,j}(t) = 2\pi i H(t) \frac{(it)^{j+k-1}}{(j+k-1)!} e^{ia_y t}.$$

Substitution of this result into (20) gives us

$$I_1 = iH(t) e^{ia_y t} \sum_{j=0}^{\infty} \frac{(-ct)^j}{j!j!} \qquad \text{and} \qquad I_2 = -H(t) e^{ia_y t} t \sum_{j=0}^{\infty} \frac{(-ct)^j}{j!(j+1)!}.$$

Now using the series representations of the Bessel functions J_0 and J_1 (see e.g. [14, Chapter 5]) we obtain

$$I_1 = iH(t) e^{ia_y t} J_0\left(2\sqrt{ct}\right) \quad \text{and} \quad I_2 = -H(t) \sqrt{\frac{t}{c}} e^{ia_\epsilon t} J_1\left(2\sqrt{ct}\right).$$

Substituting these expressions in (19) and then in (18) we arrive at the following expression for f:

$$f(t,x) = H(t)e^{iat}E\left(x\right)\left(-A\left(x\right)\sqrt{\frac{t}{c}}J_1\left(2\sqrt{ct}\right)+iB\left(x\right)J_0\left(2\sqrt{ct}\right)\right).$$

Finally we rewrite the obtained fundamental solution of the operator M in explicit form:

$$f(t,x) = H(t)\frac{e^{\frac{it}{\beta\sqrt{\varepsilon\mu}}}}{\beta\sqrt{\varepsilon\mu}}\left(\mathcal{K}_{\frac{1}{\beta}}(x)J_0\left(\frac{2\sqrt{t\left|x\right|}}{\beta\left(\varepsilon\mu\right)^{\frac{1}{4}}}\right)\right.$$
$$\left.+\frac{i\Theta_{\frac{1}{\beta}}(x)}{\beta\left(\varepsilon\mu\right)^{\frac{1}{4}}}\left(1-\frac{ix}{\left|x\right|}\right)\sqrt{\frac{t}{\left|x\right|}}J_1\left(\frac{2\sqrt{t\left|x\right|}}{\beta\left(\varepsilon\mu\right)^{\frac{1}{4}}}\right)\right).$$

Let us notice that f fulfills the causality principle requirement which guarantees that its convolution with the function from the right-hand side of (15) gives us the unique physically meaningful solution of the inhomogeneous Maxwell system (6)-(8) in a whole space.

Acknowledgement 2 *The authors wish to express their gratitute to CONACYT for the support of this work via the grant Cátedra Patrimonial No. 010286 and a Research Project.*

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